Stochastic Intensity Modelling for Structured Credit Exotics

Alexander Chapovsky, Andrew Rennie and Pedro A. C. Tavares*
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Abstract

We propose a class of credit models where we model default intensity as a jump-diffusion stochastic process. We demonstrate how this class of models can be specialised to value multi-asset derivatives such as CDO and CDO^2 in an efficient way. We also suggest how it can be adapted to the pricing of option on tranche and leverage tranche deals. We discuss how the model performs when calibrated to the market.

*2 King Edward Street, London EC1A 1HQ, U.K.; +44 (020) 7996 5937; pedro_tavares@ml.com

1 Introduction

In the last several years the market for credit derivatives experienced an explosive growth both in terms of volume and innovation. The market in standard tranche CDOs became liquid (ITX, CDX, and others) and provides possibilities for hedging correlation risk. At the same time new exotic products are traded over the counter. These can be split broadly in two categories: default derivatives with complex payoffs (bespoke tranches, CDO^2 and others) and derivatives with payoffs that depend of spread levels and mark-to-market (options on tranches, leveraged super-senior, credit CPPI, *etc*). While the former require modelling of the defaults of individual single name credits, the latter require dynamical modelling of defaults and spread levels. This growth produces a need for new models for valuing and managing the risk.

The Gaussian copula model, [1], became the industry standard for the valuation of CDO. Emergence of a skew market in CDO of standard portfolios then gave rise to a number of extensions of this model that attempt to account for the structure of observed prices (for example, [2, 3, 4, 5]). All these approaches are similar in that they model loss distribution of a basket of credits at a given time horizon starting from default probabilities of single names through some copula function. These models have had variable success in calibrating to observed prices, depending on the complexity and flexibility of the copula and some became popular. Their main advantage is that the properties of single name credits are explicit inputs, which allows one to model more complicated derivatives such as CDO^2. The main drawback is that explicit intensity dynamics is absent from these models, which makes it impossible to use these models for modelling of more exotic credit derivatives, such as option on tranches.

Recently several authors proposed models that directly model the dynamics of the loss distribution of a given portfolio, [6, 7]. Forward loss distribution is an input into the model which makes it possible to calibrate exactly to the structure of observed prices by construction. The model has dynamics as well so it possible to price options on tranches. The main drawback of this model is that single name information is not an explicit input into the model and that makes it difficult to price bespoke basket CDOs or certain types of exotic structures such as CDO^2.

Direct modelling of stochastic credit default intensities for the valuation of CDO transactions was proposed by Duffie and Garleanu, [8]. The main advantage of this approach is that the dynamics of default intensities for each credit can be specified which allows one to deal with CDO, CDO^2 and options on tranches within a single model, at least in principle. For a long time the perception was that this class of models is too complex and requires the use of Monte-Carlo methods for their implementation making efficient calibration impossible and deterring practitioners from using them in industrial applications. This motivated us to develop a modelling framework under which the default intensities of each single-name are modelled individually, yet the framework is still simple enough to allow for efficient calculations suited for use by practitioners. Very recently similar approaches were proposed in the literature, [9, 10, 11].

The remainder of the paper is organised as follows. In Section 2 we described the basic setup of the model. Then in Section 3 we discuss parametrisation and calibration of the model to the CDO tranche market. In Section 4 we extend the framework to the pricing of option-like exotic credit derivatives. Finally we conclude in Section 5.

2 Model Setup

2.1 Motivation

From practical point of view a good model should have some important features:

- Have a parametrization that is intuitive.
- Be formulated in terms of single-name credits. This allows one to account for portfolio dispersion, bespoke tranches, CDO^2 and so on in a natural way.
- Calibrate to the term structure of single-name survival probabilities easily, preferably by construction.
- Be formulated in terms of local dynamics. This allows the model, at least in principle, to price path-dependent contracts and contracts which depend non linearly on future mark-to-market such as options on tranches.
- Calibrate to standard tranches, preferably by construction. This means in particular that it should produce enough default correlation and be flexible enough to match correlation skew.
- Suggest a hedging strategy with respect to single-name intensities (*i.e.* spread leverages) and model parameters (correlation skew hedge).

The model we discuss in this paper, on one hand, has most of these features and, on the other hand, is comparable to standard copula models in terms of its computational complexity when pricing vanilla tranches.

2.2 Single credit dynamics

The basic modelling quantity in our model is stochastic intensity, λ_i, of a credit name, i. Intensity should be positive for all times

$$\lambda_i(t) > 0, \tag{1}$$

and should calibrate to the term structure of survival probabilities

$$\mathrm{E}\left[e^{-\int_0^t \lambda_i dt}\right] = p_i(t), \tag{2}$$

where $p(t)$ is implied survival probability of the credit. The following ansatz solves the single name calibration equation

$$\lambda_i(t) = \lambda_i' - \lambda_i^c(t) + \lambda_i^f(t), \tag{3}$$

where $\lambda_i'(t)$ is an auxiliary random process. Compensator, $\lambda_i^c(t)$, and forward intensity, $\lambda_i^f(t)$, are deterministic functions of time which solve equations

$$\mathrm{E}\left[e^{-\int_0^t \lambda_i' dt}\right] = e^{-\int_0^t \lambda_i^c dt}, \qquad e^{-\int_0^t \lambda_i^f dt} = p_i(t). \tag{4}$$

Term structure calibration and dynamics are separated in this ansatz: dynamics of λ_i' determines the compensator, λ_i^c, and survival probabilities, p_i, determine the forward intensities λ_i^f. However, the choice of dynamics is constrained by a requirement that total intensity stays positive for all times. In general, intensity dynamics, which satisfies (1), depends on the entire term structure of survival probabilities, $\lambda_i'[\lambda_i^f(t)]$, and is therefore different for different credits.

2.3 Multiple credit dynamics

The main object in modelling vanilla multi-name credit derivatives, such as CDO tranches, is the loss distribution of a basket of credits at various time horizons, $P(t, L)$. This loss distribution depends in general on survival probabilities of underlying credits and on correlation structure of credit defaults. In our framework intensity realizations determine default probabilities

$$P(t, L) = \mathrm{E}\left[P(t, L|\lambda_i)\right], \tag{5}$$

where expectation is taken over all possible realizations of stochastic intensities, and $P(t, L|\lambda_i)$ is a loss distribution conditioned on realized trajectories of all intensities. Defaults are independent conditional on realized intensities. Therefore, $P(t, L|\lambda_i)$ is relatively easy to calculate, since $L = \sum l_i$, where l_i are independent single name losses with distributions determined by realizations of λ_i. In this setup, if dynamics of λ_i is known for every credit, one already can in principle perform all relevant calculations. This involves calculating outer expectation in the formula above by means of Monte-Carlo simulation inside which $P(t, L|\lambda_i)$ is calculated by recursion or another simulation. The outer simulation is of very high dimension and is not suitable for efficient calculations.

2.4 Factorization of intensity dynamics

One can try to reduce the dimensionality of the problem in order to make calculations more efficient. First of all, one can observe that conditional loss distribution of the basket, $P(L|\lambda_i)$, only depends on conditional survival probabilities of the credits, and therefore only on integrals $\int_0^t \lambda_i dt$, rather then full paths, λ_i. Second, one can try to reduce dimensionality further by considering some simple low dimensional factor model for dynamics of λ_i. However, as discussed above, the need to satisfy positivity constraint for intensity, (1), makes intensity dynamics dependent on the forward survival probability curve, $\lambda_i'[\lambda_i^f(t)]$, which is different for different names. This makes the dependence structure between the integrals of intensity and therefore survival probabilities, complicated in general. This fundamental interplay between the term structure of forward survival probabilities and factor dynamics for λ_i motivates the following parametrization for the joint dynamics

$$\int_0^t \lambda_i dt = \beta_i(t) \int_0^t y dt - \phi(t, \beta_i(t)) + \int_0^t \lambda_i^f dt,$$
$$\mathrm{E}\left[e^{-u \int_0^t y dt}\right] = e^{-\phi(t,u)}, \tag{6}$$

where y is the common random intensity driver, $\beta_i(t)$ and $\phi(t, u)$ are deterministic functions of time. Functions $\beta_i(t)$ should be chosen to guarantee positivity of intensity, (1), and are therefore credit specific, as discussed above. Function $\phi^c(t, u)$ is simply a characteristic exponent of variable $X(t)$,

$$X(t) = \int_0^t y dt, \qquad \mathrm{E}\left[e^{-uX(t)}\right] = e^{-\phi(t,u)}. \tag{7}$$

To summarize, we have found a model specification, in which all conditional survival probabilities are expressed in terms of a single one dimensional variable, X,

$$p_i(t|X) = e^{-\int_0^t \lambda_i dt} = p_i(t) e^{-\beta_i(t) X + \phi(t, \beta_i(t))}, \tag{8}$$

in such a way that the term structure of implied survival probabilities, (2), is reproduced by virtue of the definition of $\phi(t, \beta_i(t))$. The positivity constraint (1) is satisfied by a credit specific choice of the functions $\beta_i(t)$, which in general depend of dynamics of X (induced by y), and term structure of the $p_i(t)$.

Expression (5) for the conditional loss distribution now reduces to

$$P(t, L) = \mathrm{E}\left[P(t, L|X)\right] = \int P(t, L|X) dP(X). \tag{9}$$

This expression demonstrates that the model has an equivalent copula formulation given by (8) and (9), with X playing the role of the central shock. The entire architecture practitioners have developed for copula models can then be employed here as well.

2.5 Note on credit correlation

The main goal of modelling the credit multi-name derivatives is to devise a mechanism that induces default correlation between different credit names. Here we would like to discuss briefly the alternatives. Defaults are modelled as a first jump of a Poisson process with some intensity and thus default of a name is correlated to its intensity by construction. Therefore, there are a number of options to induce correlations between the default events of two names:

- Default of a name causes changes in the intensity of other names;
- Default of a name causes defaults of other names directly (not affecting intensities);
- Intensity change of one name causes intensity changes of other names.

The first option corresponds to contagious default models[1], [12]. These models are attractive because they explicitly take into account the very intuitive effect of default of a name in the basket on other names in the basket. However, there is no reason why defaults of names outside of the basket should have no similar effect on the names in the basket. After all, choice of the basket is unlikely to affect real intensity dynamics of the single-names. Moreover, unless one models the entire credit universe simultaneously there are more names outside of the basket than inside. By neglecting effect of the names outside of the basket one ignores a potentially large effect.

The second option corresponds to models which allow several defaults at the same moment in time. Indeed, consider two names with correlated defaults within some time horizon. If one name defaults, intensity of other name should jump up accordingly because the other name is more likely to default now. If one insists on intensity not changing, the only way to allow non-zero default correlation is to allow for instantaneous defaults of both names. In other words, default correlation is local in time. Recently models which allow for multiple simultaneous defaults were discussed[2], [13, 14].

The third option, which we chose to follow, corresponds to the models where intensities of the credits have correlated dynamics and defaults are independent conditioned on the realization of intensity paths. Default correlations are indirectly induced by intensity correlations. It is well known that correlated intensity diffusion induces very small

[1] Gaussian copula itself can be thought of as this type of model. Indeed, it is possible to come up with a dynamics (in which all intensities of the basket decrease if there are no defaults, and jump up if one of the names in the basket defaults), which reproduces loss distributions identical to that of Gaussian copula model.

[2] Model suggested in [11] can also be viewed as belonging to this type of models. Business time of [11] used to calculate default probability experiences θ-like jumps. This is similar to intensity experiencing δ-like jumps in physical time. This means that correlation between defaults is localized in physical time.

(parametrically small) default correlations, not nearly enough to explain levels of default correlations observed in the CDO market. One needs much more extreme dynamics to obtain reasonable default correlations and is therefore forced to add correlated jumps to intensity dynamics. As was noted already by Duffie and Garleanu, [8], correlated jump-diffusion intensity dynamics induces much more default correlation. This model did not prove popular with the practitioners due to the necessity of relying on simulation methods for calculations and the availability of copula models in which semi-analytical calculations were possible. However, in this paper we discuss models with jump-diffusive intensity dynamics in which one can perform semi-analytical calculations, similar to those in copula models. More importantly, a new generation of credit derivatives (like options on tranches, LSS, credit CPPI, and others) cannot be modelled with copula models. Finally we would like to note that, at least on a finite time horizon, there is a similarity between these models and contagion models if shocks are originated by events inside as well as outside of the underlying basket. All of this makes stochastic intensity models attractive and serve as a motivation for this paper.

3 Model parametrization and Calibration

In this section we describe possible model parameterizations and corresponding calibration results. The model is described by the dynamics of y and coefficients β_i. Together they determine the distribution $P(X)$ and compensator function $\phi(t, \beta_i)$. These are sufficient to calculate the loss distribution, $P(t, L)$, and therefore prices of CDO tranches.

Note that if the dynamics of y are such that $\phi(t, u)$ is known analytically, one knows not only all conditioned survival probabilities, but also the distribution of X, because they are related by

$$\mathrm{E}\left[e^{-uX(t)}\right] = \int e^{-uX(t)} dP(X) = e^{-\phi(t,u)}. \tag{10}$$

$P(X)$ can then be found from $\phi(t, u)$ by inverse Laplace transform[3]. This makes it particularly attractive to look for models with analytical solutions for $\phi(t, u)$. Models with affine dynamics for y belong to this class and are extensively studied, [16].

3.1 Jump-only process

As discussed previously, in order to induce enough default correlation, intensities should be subject to common jumps. We start therefore with a simple jump dynamics

$$dy = j dN, \tag{11}$$

[3] Sometimes inverse Laplace transformations are seen as difficult for numerical implementation due to slow convergence. There are, however, efficient algorithms, which resolve this issue. In particular we used algorithms described in [15].

where j is the randomly distributed jump size with probability distribution $p(j)$, and dN is a Poisson process with jump intensity Λ. This is an affine model, therefore $\phi(t,u)$ is explicitly known, [16],

$$\phi(t,u) = \Lambda t \int dp(j) \left(1 - \frac{1-e^{-jtu}}{jtu}\right) \tag{12}$$

We choose coefficients $\beta_i(t)$ to be equal to *average* intensity

$$\beta_i(t) = \bar{\lambda}_i(t), \qquad \bar{\lambda}_i(t)t = \int_0^t \lambda_i^f dt. \tag{13}$$

With this choice, variable y and jumps j are dimensionless. We can also set, without loss of generality, $y(t=0)=0$.

Note that this model is already quite flexible. Indeed, one has the whole jump distribution, $p(j)$, as well as jump intensity, Λ, to calibrate CDO tranches. However, the requirement of positivity on intensity introduces bounds on $p(j)$ and Λ. In order to get some intuition about this consider flat term structure of intensity, $\lambda_i^f(t) = \lambda_i^f$. Then jump size, j, simply measures by how much intensity, λ_i, jumps relative to the forward intensity, λ_i^f. Also, in this case, $\beta_i(t) = \lambda_i^f$. Realized intensity is

$$\lambda_i(t) = \lambda_i^f y - \frac{\partial}{\partial t}\phi(t,\lambda_i^f) + \lambda_i^f, \tag{14}$$

where

$$\frac{\partial}{\partial t}\phi(t,u) = \Lambda \int dp(j) \left(1 - e^{-jtu}\right). \tag{15}$$

For $jtu \ll 1$ this function grows linearly, $\partial_t \phi(t,u) \sim t$, and then saturates exponentially at $jtu \gg 1$, $\partial_t\phi(t,u) \to \Lambda$. As y is always positive $\lambda_i^f - \partial_t\phi(t,\lambda_i^f)$ should be positive to guarantee that $\lambda_i(t)$ is positive. This is guaranteed for at any time and for any jump size if

$$\Lambda < \lambda_i^f. \tag{16}$$

This is too restrictive in practice because common jump intensity, Λ, is constrained by the intensity of the tightest credit, which can be very small, resulting ultimately in very small default correlations. A more flexible bound is obtained if one constrains intensity to be positive only during the finite time, t, of the order of the maturity of the trade[4]. In that case it is enough if

$$jt\Lambda \ll 1 \tag{17}$$

as in this limit

$$\frac{\partial}{\partial t}\phi(t,u) = \Lambda t u \int \mathrm{d}p(j) j. \tag{18}$$

[4]It is not unreasonable to do this because, in practice, intensity term structure is upwards sloping, relaxing the constraint for longer maturities

Table 1: Jump-only calibration of 5y DJIG on 20 Sep 2005 ($\Lambda = 100\%$)

Attach	Detach	Bid	Offer	Mid	Model
15%	30%	0.07%	0.08%	0.07%	0.07%
10%	15%	0.14%	0.16%	0.15%	0.14%
7%	10%	0.27%	0.3%	0.29%	0.29%
3%	7%	1.21%	1.24%	1.23%	1.21%
0%	3%	42.38%	42.88%	42.63%	42.39%

Jump	Prob
0.27	0.212
0.50	0.085
5.50	0.005
10.00	0.001
Syst.	0.0007

Table 2: Jump-only calibration of 5y ITX on 20 Sep 2005 ($\Lambda = 100\%$)

Attach	Detach	Bid	Offer	Mid	Model
12%	22%	0.06%	0.07%	0.06%	0.06%
9%	12%	0.1%	0.13%	0.11%	0.13%
6%	9%	0.21%	0.26%	0.24%	0.21%
3%	6%	0.76%	0.8%	0.78%	0.79%
0%	3%	24.5%	25.5%	25.%	24.55%

Jump	Prob
0.27	0.410
0.55	0.399
8.00	0.003
10.03	0.001
Syst.	0.0005

Note that this bound does not depend on forward, λ_i^f. This bound is already quite flexible. For example, for a 10 year horizon it allows ten-fold intensity jumps with 0.1% intensity, or two-fold jumps with 5% intensity. Smaller, more frequent jumps can be outside of this bound. The main reason for the bound is the linear dynamics for intensity. The size of the jumps depends on the forward rather than on the current level of intensity. These small jumps, however, can be effectively described by diffusion, suggesting an extension to the model which we describe below.

Another problem might occur in presence of very large rare jumps. If intensity of these jumps is small enough no intensity bounds are violated, so there is no problem in principle. However, there might be a technical problem because very large jumps force one to calculate $P(X)$ for very large values of X, slowing down the calculation. However, very large jumps of intensity mean that intensity widens so much that names default very quickly. So an effective description of this behavior is to include a systemic default of all names with some intensity, λ_{sys}. One has to keep in mind that this is just an effective description of very large jumps, which is convenient to make calculations more efficient, but it is not needed in principle. An obvious constraint applies to λ_{sys}

$$\lambda_{\text{sys}} < \lambda_i^f. \tag{19}$$

Calibration to index tranche market values is shown in Tables 1 and 2. The model is able to fit the observed 5 year prices in both CDX and ITX markets very accurately. the results also exhibit three distinct jump scales: a small (<1) group of jumps, a second

Table 3: Jump-CIR calibration of 5y ITX on 3 Nov 2005 ($\theta = \eta = 1$, $\sigma = 35\%$, $\Lambda = 3\%$)

Attach	Detach	Bid	Offer	Mid	Model
12%	22%	0.05%	0.07%	0.06%	0.06%
9%	12%	0.1%	0.13%	0.12%	0.11%
6%	9%	0.21%	0.24%	0.23%	0.22%
3%	6%	0.91%	0.94%	0.93%	0.91%
0%	3%	29.%	29.5%	29.25%	29.27%

Jump	Prob
2.10	0.65
7.00	0.1
Syst.	0.00053

Table 4: Jump-CIR calibration of 10y ITX on 3 Nov 2005 ($\theta = \eta = 1$, $\sigma = 35\%$, $\Lambda = 1\%$)

Attach	Detach	Bid	Offer	Mid	Model
12%	22%	0.2%	0.25%	0.23%	0.25%
9%	12%	0.46%	0.56%	0.51%	0.45%
6%	9%	0.96%	1.06%	1.01%	1.%
3%	6%	5.05%	5.35%	5.2%	5.09%
0%	3%	55.75%	57.25%	56.5%	56.09%

Jump	Prob
2.10	0.65
7.00	0.1
Syst.	0.0019

group of jumps of the order $1 < j < 10$ and finally a group of very large jumps, which for numerical reasons were replaced with a systemic component, as was described earlier. The fact that this structure arises naturally in calibrating the model is both interesting and reassuring because several authors have observed (see [4, 13, 14]) that models with 3-dimensional parameter structures are successful at recovering market prices.

3.2 Jump-CIR process

In calibrating the jump-only process we observed that jumps fall into groups according to their size. In particular we noted that a group of small jumps ($j < 1$) arises in the calibration. This *fine structure* is similar to diffusion so it is natural to describe small frequent jumps by adding a nonlinear diffusion to the dynamics of y:

$$y = y_D + y_J, \qquad dy_D = \theta(\eta - y_D)dt + \sigma\sqrt{y_D}dW, \qquad dy_J = jdN. \tag{20}$$

Here, affine diffusive dynamics is chosen to preserve tractability of $\phi(t, u)$, [8, 16].

We again show two example of calibration in Tables 3 and 4 but now focus on comparing 5 and 10 year trades to highlight certain features. The two calibrations are obtained separately with time-constant parameters but we ensured that all parameters except the jump intensity is held constant. While it is possible to obtain a more accurate calibration if we free the parameters from this restriction, the purpose here is to ensure as consistent a parametrisation across market prices as possible, especially with respect to the term structure properties of the model. We concluded this could be achieve by changing only the jump intensity and systemic intensity. Given these considerations we observe that calibration is accurate but between the 5 and 10 year maturity we see a very sharp drop in jump intensity. This fact suggests the next generalisation: to add mean reversion to the jump process to improve the term structure properties of the model.

3.3 Non-linear jump-diffusion process

Alternative specification of the model, which includes mean reversion of jumps, with analytic solution for $\phi(t, u)$ is canonic affine parametrization like in [8, 9, 10]:

$$dy = \theta(\eta - y)dt + \sigma\sqrt{y}dW + jdN, \tag{21}$$

where jumps are exponentially distributed and so $p(j)$ is parametrised by just one parameter. This choice provides less flexibility in calibrating to CDO tranches and so we generalised it further to account for any distribution of j. The cost of such extension is that the compensator is no longer analytic and we can no longer apply the inverse Laplace method. Instead the distribution $P(X)$ is determined by solving for

$$dy(t) = \mu(t, y)dt + \sigma(t, y)dW + jdN \tag{22}$$

$$dX(t) = y(t)\,dt\ . \tag{23}$$

where μ and σ are generic functions of time and process y. These can be defined by (21) but that is not strictly necessary. This is a very rich parametrisation which can feature volatility smile, mean reversion and term structure of all parameters. As a result the calculation is more complex. A 2-dimensional PIDE needs to be solved to determine $P(X)$. Once $P(X)$ is known, the compensator can be determined through (10).

Calibration results are shown in Tables 5 and 6. As before we aimed to calibrate the model with minimal term structure. In particular we keep all parameters constant expect jump and systemic intensities. We set $\mu(t, y) \equiv \mu(y) = \theta(y)(\eta - y)$ and $\sigma(t, y) \equiv \sigma(y)$. On average $\bar{\theta} \sim 1$ and $\bar{\sigma} \sim 30\%$. Spectrum of jumps remains constant in time. As expected the mean reversion effect on jumps reduces the term structure of calibrated jump intensity. It is also noticeable that calibration quality is worse than in earlier cases. Partially this is due to the fact that we sacrificed quality of fit for the sake of flatter term structure. Allowing for a steeper term structure, including spectrum of jumps will improve the calibration quality. In addition the model dynamics is now highly non-linear and therefore calibration procedure itself is now more subtle. More advanced methods of calibration are appropriate in this case.

Table 5: Non-linear calibration of 5y ITX on 3 Nov 2005 ($\eta = 1$, $\bar{\theta} \sim 1$, $\bar{\sigma} \sim 30\%$, $\Lambda = 1.00\%$)

Attach	Detach	Bid	Offer	Mid	Model
12%	22%	0.05%	0.07%	0.06%	0.05%
9%	12%	0.1%	0.13%	0.12%	0.11%
6%	9%	0.21%	0.24%	0.23%	0.24%
3%	6%	0.91%	0.94%	0.93%	0.9%
0%	3%	29.%	29.5%	29.25%	29.27%

Jump	Prob
1.02	0.25
3.15	3.50
4.30	1.30
6.49	0.8
Syst.	0.00053

Table 6: Non-linear calibration of 10y ITX on 3 Nov 2005 ($\eta = 1$, $\bar{\theta} \sim 1$, $\bar{\sigma} \sim 30\%$, $\Lambda = 0.75\%$)

Attach	Detach	Bid	Offer	Mid	Model
12%	22%	0.2%	0.25%	0.23%	0.26%
9%	12%	0.46%	0.56%	0.51%	0.45%
6%	9%	0.96%	1.06%	1.01%	1.1%
3%	6%	5.05%	5.35%	5.2%	4.91%
0%	3%	55.75%	57.25%	56.5%	57.92%

Jump	Prob
1.02	0.25
3.15	3.50
4.30	1.30
6.49	0.8
Syst.	0.0019

3.4 Idiosyncratic intensity dynamics

Discussion so far was based on one factor model for intensity dynamics. That means that the intensity of all credits changes collectively. One might find this absence of idiosyncratic dynamics unnatural. It is easy to extend the model to allow for idiosyncratic dynamics by introducing credit specific factors. Schematically

$$\lambda_i = (\beta_i y - y^c) + (y_i - y_i^c) + \lambda_i^f, \tag{24}$$

where $\lambda_i^f(t)$ is forward, and y and y_i are common and idiosyncratic drivers respectively with their compensators

$$\mathrm{E}\left[e^{-\beta_i \int_0^t y dt}\right] = e^{-\int_0^t y^c dt}, \qquad \mathrm{E}\left[e^{-\int_0^t y_i dt}\right] = e^{-\int_0^t y_i^c dt}. \tag{25}$$

This is a much more general specification then we considered previously and calculations are more complicated in this case. However, it is natural to think that prices of CDO tranches should be determined by collective rather then idiosyncratic dynamics. To see how this intuition manifests itself in our setup let us consider loss distribution of a credit basket

$$P(t, L) = \mathrm{E}\left[P(t, L|y, y_i)\right] = \mathrm{E}\left[\mathrm{E}\left[P(t, L|y, y_i)\right]|y\right], \tag{26}$$

where outer expectation is over y, and inner expectation is over y_i conditioned on y. Observe that, due to conditional independence of defaults, all terms in $P(t, L|y, y_i)$ are proportional to either conditional survival or conditional default probability of credit i. Therefore $P(t, L|y, y_i)$ is linear in $\exp(-\int_0^t (y_i - y_i^c)dt)$. By definition of y_i^c expectation of this quantity is 1, which means that loss distribution, $P(t, L)$, and CDO tranche prices do not depend on idiosyncratic dynamics. This argument is not completely correct because it neglects indirect effect through positivity of intensity constraint. Through this constraint idiosyncratic dynamics can have an effect on allowed values of credit's coupling to common driver β_i, and thus indirectly affect CDO tranche prices. If one assumes that constraint is satisfied, idiosyncratic effect completely drops out.

4 Application to structured credit exotics

As discussed above, the stochastic intensity model described in this paper induces default copula and therefore allows one to model credit derivatives with payoffs dependent on the defaults of underlying credits (like, for example, CDO or CDO^2 tranches) by semi-analytic calculations or conditioned Monte-Carlo similar to conventional copula models. The model is also formulated in terms of the local dynamics of intensity of each name and therefore allows modelling of credit derivatives whose payoffs depend on the paths of loss and intensity, like CPPI on credit index, by direct simulation in the same self consistent model. In this section we consider the pricing of other types of exotic credit derivatives whose value depend in a nonlinear way on the mark-to-market value of other underlying credit derivatives. Derivatives of this type are options on CDO tranches, leveraged super-senior tranches with various triggers (loss, intensity and mark-to-market triggers), CDO tranches with counterparty risk, where counterparty credit risk is correlated with the credit risk of the underlying names, and so on. It is desirable to price these exotic derivatives in the same model as vanilla derivatives in order to achieve consistency and avoid arbitrage.

4.1 Approximating model dynamics

The main difficulty in modelling products of this type is a combination of very high dimension and a need to calculate the value of underlying derivatives potentially at every time step and for every realized value of market variables. One method to deal with this type of problems is the often called *American Monte-Carlo*, [17], which relies on trying to approximately estimate the values of a relevant contract as a function of smaller number of variables while running the simulation. Here we try to approach the problem differently. We try to find a low dimensional model, which approximates the original high dimensional model. We then solve the pricing problem in a low dimensional approximating model by backward induction methods.

Consider a derivative with a large basket of credits as underlying. The price of this derivative at time t depends on realized defaults and realized term structure of intensities of the underlying credits at that time

$$V_t = V_t\left(I_t^{(i)}, \lambda_t^{(i)}\right), \tag{27}$$

where $I_t^{(i)}$ are default indicators and the $\lambda_t^{(i)}$ denote the implied term structure of survival probabilities. It satisfies equation

$$\frac{1}{B_t}\, V_t = \frac{1}{B_{t+\delta t}} \mathrm{E}\left[F(V_{t+\delta t}) | I_t^{(i)}, \lambda_t^{(i)}\right] \tag{28}$$

where F is some nonlinear function. This is a backward induction equation of very high dimension. We will try to approximate this pricing problem by a low dimensional pricing problem which is more tractable. In doing so it is crucial to find factors which approximate the problem well. As discussed in previous section, in our framework current intensities of all credits in the basket are determined by a single common driver, y, which follows jump diffusion. We assume that the loss distributions of the basket, $P(t, L)$, are calculated using the appropriate methods of Sections 2 and 3. In particular, loss distribution has representation

$$P(t, L) = \mathrm{E}\left[P(t, L|X)\right] = \int P(t, L|X) dP(X), \tag{29}$$

where variable X,

$$X_t = \int_0^t y dt, \tag{30}$$

is connected to the realized survival probability up to time t, and $P(t, L|X)$ is the loss distribution at time t conditioned on realized survival probabilities.

What should one choose as a minimal set of factors for the low dimensional approximating model? Any derivative on the credit basket must depend at least on the current level of intensities and realized loss. That is why y, corresponding to current intensities, and L, corresponding to realized loss, must be included in the minimal set of factors. Additionally, it is important to require that the approximating pricing model reproduces the loss distributions of the basket (and therefore all tranche prices), as implied by the full model, by construction. In order to achieve that we have to include X, corresponding to realized survival probabilities, in the minimal set of factors for our approximating model. In this way we will reproduce also conditioned loss distributions, $P(t, L|X)$, which can serve as the motivation to include X as a factor in the approximating model in its own right. Loss distribution then depends on realized survival probabilities, and thus on X, which in turn is correlated with the current level of intensity, y.

To summarize, our approximating model has three factors: y, L and X, with some associated dynamics induced by the full model, which we discuss below. Note, that information about individual credits does not feature in the approximated model explicitly. Instead, it enters implicitly through the loss distributions, which model dynamics has to reproduce. This makes our approximating model similar, in spirit, to dynamical loss models of [6, 7] in that it is trying to model loss of the basket as a dynamic variable.

In this low dimensional model the price of derivatives depends on the three factors discussed above

$$V_t = V_t(y_t, X_t, L_t), \tag{31}$$

and satisfies the following tree dimensional backward induction equation

$$\frac{1}{B_t}\, V_t = \frac{1}{B_{t+\delta t}} \mathrm{E}\left[F(V_{t+\delta t})|y_t, X_t, L_t\right]. \tag{32}$$

Note that the price of derivatives in this model depends on variable X, which is connected to the realized survival probabilities. This may seem unnatural and even incorrect because intuitively the price should depend on current and future credit intensities, not past intensities. This seeming paradox is resolved as follows. Recall that we needed to include X as one of factors in order to reproduce loss distributions. These carry information about dispersion of credits in the basket. Imagine now that by the time t some assets defaulted. If credits in the basket are different, the value of the derivative depends on identities of defaulted credits. Imagine now that the only information available is the total loss up to the time t. To price the derivative one now has to assess which credits are more likely to have defaulted and which credits are more likely to remain in the basket. In order to do so one needs to know the past realized intensities, and therefore, the price of the derivative does indeed depend on X. In other words, X effectively captures dispersion of the credits in the basket.

To complete the specification of the model one needs to specify the dynamics of the drivers y, X and L. Dynamics of y is given by the same jump-diffusion process as that in the full model, (22),

$$dy_t = \mu_t(y)dt + \sigma_t(y)dW + j_t(y)dN, \tag{33}$$

where we changed notation slightly in order to make it consistent with this Section. Note, that μ, σ can be functions of t and y, and j is a random variable with probability distribution, $p(j)$, which also can depend on t and y. Like in the full model, variable X follows

$$dX = ydt. \tag{34}$$

Finally loss, L, is a dynamical variable, with dynamics chosen to calibrate conditional loss distributions $P(t, L|X)$ for all t and X. We will find this transition probability neglecting, here and in the remainder of this Section, terms of higher orders in δt. Transition probability satisfies

$$P(t+\delta t, L_2|X_2) = \int dL_1 P(t+\delta t, L_2|t, L_1, X_1, y_1) P(t, L_1|X_1), \tag{35}$$

where X_1, L_1 and y_1 are state variables at time t, X_2 and L_2 are state variables at time $t + \delta t$, and $X_2 = X_1 + y_1 \delta t$. We need to solve this equation to obtain local loss transition probabilities, $P(t + \delta t, L_2 | t, L_1, X_1, y_1)$, which determine the local dynamics of loss variable, L, consistent with conditional loss distributions, $P(t, L|X)$. Similar calculation is performed in [7], where loss transition rates are calculated for unconditional loss distribution. We look for a kernel in the following form

$$P(t + \delta t, L_2 | t, L_1, X_1, y_1) = \delta(L_1 - L_2) + \delta t \Lambda(L_2 | t, L_1, X_1, y_1), \tag{36}$$

where $\Lambda(L_2 | t, L_1, X_1, y_1)$ is a transition intensity from L_1 to L_2. It should satisfy

$$\Lambda(L_2 | t, L_1, X_1, y_1) = 0, \qquad L_2 < L_1, \tag{37}$$

due to positivity of the loss and

$$\int dL_2 \, \Lambda(L_2 | t, L_1, X_1, y_1) = 0, \tag{38}$$

due to probability conservation. Λ satisfies an integral-differential equation,

$$\left(\frac{\partial}{\partial t} + y \frac{\partial}{\partial X} \right) P(t, L|X) = \int dL' \Lambda(L|t, L', X, y) P(t, L'|X). \tag{39}$$

In practice loss distributions, $P(t, L|X)$, are discrete in L. This means that L is a discrete variable, $0 \leq L \leq L_{\max}$. In discrete setting loss densities become vectors of loss probabilities, $P_L(t, X)$. The integral equations above become matrix equations,

$$P_{L_2}(t + \delta t, X_2) = \sum_{L_1=0}^{L_{\max}} \left(\delta_{L_2 L_1} + \delta t \; \Lambda_{L_2 L_1}(t, X_1, y_1) \right) P_{L_1}(t, X_1), \tag{40}$$

where $P_L(t, X_2) = P(t, L|X)$ are vectors of loss probabilities, and $\Lambda_{L_2 L_1}(t, X, y)$ are matrices of transition probabilities, which we need to find, satisfying

$$\begin{gathered} \Lambda_{L_2 L_1}(t, X_1, y_1) = 0, \qquad L_2 < L_1, \\ \sum_{L_2=0}^{L_{\max}} \Lambda_{L_2 L_1}(t, X_1, y_1) = 0. \end{gathered} \tag{41}$$

An additional constraint is that loss cannot be bigger then the maximum value, $L_{\max}$,

$$\Lambda_{L_{\max} L_{\max}}(t, X, y) = 0. \tag{42}$$

Therefore, transition probabilities $\Lambda_{L_2L_1}$ in general have the form

$$\Lambda_{L_2L_1} = \begin{pmatrix} \Lambda_{00} & 0 & \dots & 0 & 0 \\ \Lambda_{10} & \Lambda_{11} & \dots & 0 & 0 \\ \Lambda_{20} & \Lambda_{21} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_{L_{\max}-1,0} & \Lambda_{L_{\max}-1,1} & \dots & \Lambda_{L_{\max}-1,L_{\max}-1} & 0 \\ \Lambda_{L_{\max},0} & \Lambda_{L_{\max},1} & \dots & \Lambda_{L_{\max},L_{\max}-1} & 0 \end{pmatrix} \tag{43}$$

The solution of (40) is not unique. One needs to provide additional constraints to a find a unique solution. Following [7], we look for solution with bi-diagonal matrix $\Lambda_{L_2L_1}$

$$\Lambda_{L_2L_1} = \begin{pmatrix} -\Lambda_0 & 0 & \dots & 0 & 0 \\ \Lambda_0 & -\Lambda_1 & \dots & 0 & 0 \\ 0 & \Lambda_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -\Lambda_{L_{\max}-1} & 0 \\ 0 & 0 & \dots & \Lambda_{L_{\max}-1} & 0 \end{pmatrix} \tag{44}$$

This ansatz means that we are only looking for solutions with localized loss transition probabilities. This is consistent with the picture of just one default per time tick[5]. Constrained in this way transition probabilities are unique and are provided by solution

$$\Lambda_L(t, y, X) = -\frac{1}{P_L(t, X)} \sum_{L'=0}^{L} \frac{1}{\delta t} \left(P_{L'}\left(t + \delta t, X + y\delta t\right) - P_{L'}\left(t, X\right) \right). \tag{45}$$

Alternatively, the same solution can be written as

$$\Lambda_L(t, y, X) = \frac{1}{P_L(t, X)} \sum_{L'=L_1+1}^{L_{\max}} \frac{1}{\delta t} \left(P_{L'}\left(t + \delta t, X + y\delta t\right) - P_{L'}\left(t, X\right) \right), \tag{46}$$

This completes specification of the approximating model for options-like contracts.

4.2 Pricing of derivatives

Given the dynamics of the model a derivative contract, $V(t, y, X, L)$, can be priced by backward induction equation

$$V(t - \delta t, y, X, L) = F\left[\frac{B_{t-\delta t}}{B_t} \left(1 + \delta t \mathcal{L}\right) V(t, y, X, L) + f(t, y, X, L) \right], \tag{47}$$

[5] Such processes are often called *pure-birth processes*, as described in [18].

where $B_{t-\delta t}/B_t$ is the usual discounting factor, operator $\mathcal{L}$ describes dynamics of the model, $f(y, X, L)$ is the source term which describes the contract, and function F describes any non-linear early exercise conditions of the contract. Operator $\mathcal{L}$ describing model dynamics is given by

$$\begin{aligned}\mathcal{L}V(t, y, X, L) &= \left(\sigma^2 \frac{\partial^2}{\partial y^2} + \mu \frac{\partial}{\partial y} + y \frac{\partial}{\partial X}\right) V(t, y, X, L) \\ &+ \Lambda \int dP(j) \left(V(t, y + j, X, L) - V(t, y, X, L)\right) \\ &+ \Lambda_L(t, y, X) \left(V(t, y, X, L + 1) - V(t, y, X, L)\right), \end{aligned} \tag{48}$$

where Λ_L is loss transition probability, Λ is intensity of (default intensity) jumps and $P(j)$ is distribution of (default intensity) jumps. The first term describes diffusion and drifts, second term describes intensity jumps and the third term describes loss transitions.

A contract is specified by its value at expiry, $V(T, y, X, L)$, the source term which describes the the cash flows of the contract, $f(t, y, X, L)$, and early exercise function $F[V(t, y, X, L)]$. Below we give several examples of contract specifications in this model.

4.2.1 Vanilla tranches

To price vanilla tranche one needs to price separately default leg and coupon leg. Default leg is given by pricing a contract in our model with the following specification

$$\begin{aligned} V(T, y, X, L) = 0, \qquad F[V] = V, \\ f(t, y, X, L) = \delta t \ \Lambda_L(t, y, X, L) \ \left(f_T(L + \Delta L) - f_T(L)\right), \end{aligned} \tag{49}$$

where T is the maturity of the trade and $f_T(L)$ is the tranche loss function

$$f_T(L) = \max(\min(L, L_E) - L_A, 0), \tag{50}$$

where L_A and L_E are attachment and exhaustion of the tranche respectively.

Coupon leg is given by the following choices

$$\begin{aligned} V(T, y, X, L) = 0, \qquad F[V] = V, \\ f(t, y, X, L) = c \ \delta t \ (L_E - L_A - f_T(L)), \end{aligned} \tag{51}$$

where c is the coupon and $f_T(L)$ is the tranche loss function, the same as above. This does not include amortization of the tranche with recovery, which has to be included for super senior tranches.

4.2.2 European option on tranche

Option on tranche is a contract where one has a right to buy underlying tranche, V_T, at exercise date T at coupon level, c. Option on tranche is given by the following specification

$$V(T,y,X,L) = \max(V_T(T,y,X,L),0), \qquad F[V] = V,$$
$$f(t,y,X,L) = 0. \tag{52}$$

4.2.3 Leveraged tranche

In leveraged tranche contract the buyer receives coupon, c, on notional of the contract, N, until one of the triggers is hit. After that the buyer has an option to choose between losing his collateral, L_{cap}, and holding the underlying tranche, V_T, which corresponds to de-leveraging. The contract is specified by

$$V(T,y,X,L) = 0, \qquad f(t,y,X,L) = c\ \delta t\ N,$$
$$F[V] = (1 - I(t,y,X,L))\ V + I(t,y,X,L)\ \max(V_T, -L_{\text{cap}}), \tag{53}$$

where $I(t,y,X,L)$ is trigger indicator

$$I(t,y,X,L) = \begin{cases} 1, & \text{if trigger is hit} \\ 0, & \text{if trigger is not hit} \end{cases} \tag{54}$$

We see that early exercise function is not just a function of the value of the derivative, V, but it is also a function of value of underlying tranche, V_T. Trigger indicator can depend on t,y,X,L as well as on other underlying contracts.

Let us now discuss the triggers on leveraged tranche. Usually one distinguishes between loss triggers, index spread triggers and mark-to-market of the underlying tranche triggers. It is clear how to define trigger indicator function in case of loss trigger. In case of mark-to-market triggers indicator function simply becomes a function of V_T. In case of index spread triggers, trigger indicator function depends additionally on the index spread of the underlying credit basket corresponding to the current state, $c_{\text{index}}(t,y,X,L)$,

$$c_{\text{index}}(t,y,X,L) = \frac{V_{\text{index default leg}}(t,y,X,L)}{V_{\text{index coupon leg}}(t,y,X,L)}. \tag{55}$$

$V_{\text{index default leg}}$ and $V_{\text{index coupon leg}}$ are default and coupon leg (with 100% coupon) of the underlying index, which can be calculated on the same lattice as described in Section 4.2.1.

To summarize, to price leveraged tranche one needs to first calculate value of the underlying tranche, V_t, value of the index default and coupon legs, $V_{\text{index default leg}}$ and

$V_{\text{index coupon leg}}$, for every value of t, y, X, L. Then one can calculate leverage tranche value, V, on the same lattice, using specification (53), with trigger indicator function in general dependent on L, V_T, $V_{\text{index default leg}}$ and $V_{\text{index coupon leg}}$.

4.2.4 Tranche with counterparty risk

Tranche with counterparty default risk, V, is a derivative with vanilla tranche, V_T as underlying. It is given by the following specification

$$V(T, y, X, L) = V_T(T, y, X, L), \qquad f(t, y, X, L) = f_T(t, y, X, L),$$

$$F[V] = (1 - \delta t \lambda_c)V + \delta t \lambda_c F_{\text{pay-on-default}}[V_T], \tag{56}$$

where λ_c is intensity of the counterparty default corresponding to the current state, $\lambda_c = \lambda_c(t, y, X, L)$, and $F_{\text{pay-on-default}}$ is the payment on default of the counterparty, which depends on the current mark-to-market of the underlying tranche, V_T. Two possible choices for $F_{\text{pay-on-default}}$ are

$$F_{\text{pay-on-default}} = 0, \tag{57}$$

which correspond to no payments, and

$$F_{\text{pay-on-default}} = \min(V_T, L_{\text{cap}}), \tag{58}$$

which corresponds to counterparty paying mark-to-market of the underlying tranche only up to a certain cap L_{cap}. What remains is to determine the intensity of the counterparty default, λ_c, corresponding to the current state. As discussed in Section 2 survival probability conditioned on realized intensities depends only on variable X, $p_c(t, y, X, L) = p_c(t|X)$, and is given by

$$p_c(t|X) = e^{-\int_0^t \lambda_i dt} = p_c(t) e^{-\beta_c(t)X + \phi(t, \beta_c(t))}, \tag{59}$$

$$\beta_c(t)t = \int_0^t \lambda_c^f dt, \tag{60}$$

where $p_c(t)$ is implied survival probability, and λ_c^f is the forward intensity corresponding to $p_c(t)$. Similarly to the calculate of the loss transition probabilities in Section 4.1, we can calculate the intensity of the counterparty default, λ_c

$$\lambda_c(t, y, X, L) = -\frac{1}{p_c(t|X)} \frac{1}{\delta t} \left(p_c(t + \delta t | X + y\delta t) - p_c(t|X) \right). \tag{61}$$

This completes the definition of the problem and tranche with counterparty risk can be priced by backward induction.

5 Conclusions

In this paper we discussed a stochastic intensity model in the context of pricing exotic structured credit derivatives. The model is defined in terms of the microscopic local dynamics of intensities of individual names and therefore can be used for pricing a wide range of derivatives. We discussed various parameterisations of the model with a view to find parameterisations, which are as economical as possible, while still containing relevant degrees of freedom to be useful for pricing. From the point of view of vanilla tranche pricing the model induces one-factor copula for individual credit defaults. This provides a bridge between stochastic intensity modelling and more conventional default modelling with copulas. This also allows a variety of techniques developed for copula models to be used for the stochastic intensity model discussed here.

Efficient calculation of vanilla tranches allows one to attempt a brute force calibration of the model dynamics to the standard tranche CDO market. We find that the model, even in its simplest specification, contains relevant degrees of freedom and is flexible enough to calibrate well to separate maturities. More sophisticated parametrisations of the model dynamics allows one to calibrate reasonably well to the standard tranches of different maturities and in this way effectively model the term structure of the correlation skew. High-precision calibration to different instruments with different maturities will benefit from more advanced techniques.

We also discussed the pricing of various exotic derivative contracts in the framework of stochastic intensity models. As the model is specified microscopically for individual credits many contracts can be priced by Monte-Carlo simulation. Option-like contracts, which require backward induction for pricing, can be priced in a low-dimensional effective model for loss with dynamics induced by the full stochastic intensity model.

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